Math 246C Lecture 18 Notes

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1 Analyticity, Maximum Principle, and Hartogs' Lemma

1.1 Analyticity of holomorphic functions

Last time, we defined holomorphic functions of several complex variables: if $\Omega \subseteq \mathbb{C}^n$ is open, then $f \in \mathrm{Hol}(\Omega)$ if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial \overline{z}_j} = 0$ for all j.

Theorem 1.1. *Let $D \subseteq \mathbb{C}^n$ be a polydisc centered at 0, and let $f \in \mathrm{Hol}(D)$. We have, with normal convergence in D:*

$$f(z) = \sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha.$$

Here, normal convergence means that $\sum u_j$ converges normally in Ω ($\sum \sup_K |u_j| < \infty$) for all compact $K \subseteq \Omega$.

Proof. Let $D' = \{|z_j| < r_j'\}$ for $1 \leq j \leq n$, where $0 < r_j' < r_j$ (and $D = D_1 \times \cdots \times D_n$, $D_j = \{|z_j| < r_j\}$). Then, by Cauchy's integral formula,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D'} \frac{f(\zeta)}{(\zeta - z)^E \, d\zeta, \qquad E = (1, \dots, 1}.$$

If $|\zeta_j| = r_j'$ and $|z_j| \leq r_j'' < r_j'$, then

$$\frac{1}{\zeta_j - z_j} = \frac{1}{\zeta_j} \sum_{k=0}^\infty \left(\frac{z_j}{\zeta_j} \right)^k.$$

Then

$$\frac{1}{(\zeta - z)^E} = \sum_{\alpha \in \mathbb{N}^n} \frac{z^\alpha}{\zeta^{\alpha + E}}, \qquad (\zeta, z) \in \partial_0 D' \times D'$$

with normal convergence. We get

$$f(z) = \sum_\alpha z^\alpha \frac{1}{(2\pi i)^n} \int_{\partial_0 D'} \frac{f(\zeta)}{\zeta^{\alpha+E}} \, d\zeta = \sum_\alpha z^\alpha \frac{\partial^\alpha f(0)}{\alpha!}.$$

As $\overline{D'} \subseteq D$ is arbitrary, the result follows. $\square$

Corollary 1.1. *Let $\Omega \subseteq \mathbb{C}^n$ be open and connected. If $f \in \mathrm{Hol}(\Omega)$ and $\partial^\alpha f(z_0) = 0$ for all $\alpha \in \mathbb{N}^n$ for some $z_0 \in \Omega$, then $f \equiv 0$.*

Proof. The proof is the same as for the 1-dimensional case. $\square$

1.2 The maximum principle

Theorem 1.2 (maximum principle)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be open and connected. If $f \in \mathrm{Hol}(\Omega)$ and $|f|$ assumes a local maximum in Ω, then f is constant.*

Proof. Let $z_0 \in \Omega$ be such that $|f(z_0)| \geq |f(z)|$ for all z in a neighborhood of z_0. Let $r > 0$ be such that $\{|z - z_0| < r\} \subseteq \Omega$, and consider $g_a(\tau) = f(z_0 + a\tau)$, where $a \in \mathbb{C}^n$ with $|a| = 1$ and $|\tau| < r$. Then $g_a \in \mathrm{Hol}(|\tau| < r)$, and $|g_a|$ has a local maximum at 0. So $g_a(\tau) = g_a(0)$ in $|\tau| < r$ by the maximum principle for $\mathbb{C}$. Since a is arbitrary, we get $f(z) = f(z_0)$ in $|z - z_0| < r$. By the previous corollary, $f = f(z_0)$ in Ω. $\square$

1.3 Hartogs' lemma

We will prove the following theorem.

Theorem 1.3 (Hartogs' theorem on separately holomorphic functions)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be open, and let $u : \Omega \to \mathbb{C}$ be separately holomorphic (holomorphic in each variable z_j, when the other variables are kept fixed). Then $u \in \mathrm{Hol}(\Omega)$.*

Remark 1.1. We do not even assume that u is measurable.

Remark 1.2. The corresponding result in the real domain is not true: for

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0), \end{cases}$$

$x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ are real analytic, but f is not continuous at $(0,0)$ (let alone differentiable).

Here is our starting point.

Proposition 1.1 (Hartogs' lemma)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let (u_j) be subharmonic in Ω such that for all compact $K \subseteq \Omega$, there exists an M_K such that $u_j(z) \leq M_K$ for all $z \in K$ and $j = 1, 2, \ldots$. Assume that there is a $C < \infty$ such that for all $z \in \Omega$*

$$\limsup_{j \to \infty} u_j(z) \leq C.$$

Then for every compact set $K \subseteq \Omega$ and each $\varepsilon > 0$, there exists an N such that for all $j \geq N$,

$$u_j(z) \leq C + \varepsilon, \qquad z \in K.$$

Proof. Replacing Ω by a relatively compact domain containing K, we can assume that (u_j) is bounded above in Ω or even that $u_j \leq 0$ in Ω. Given compact $K \subseteq \Omega$, let $0 < r < \operatorname{dist}(K, \Omega^c)/3$ and recall the sub-mean value property:

$$u_j(z) \leq \frac{1}{\pi r^2} \iint_{|z-\zeta| \leq r} u_j(\zeta)\, d\lambda(\zeta), \qquad z \in K.$$

By Fatou's lemma,

$$\limsup_{j \to \infty} \iint_{|z-\zeta| \leq r} u_j(\zeta)\, d\lambda(\zeta) \leq \iint_{|z-\zeta| \leq r} \limsup_{j \to \infty} u_j(\zeta)\, d\lambda(\zeta) \leq C\pi r^2.$$

Thus, for all $z \in K$, there exists j_z such that if $j \geq j_z$, then

$$\iint_{|z-\zeta| \leq r} u_j(\zeta)\, d\lambda(\zeta) \leq \pi r^2 (C + \varepsilon/2).$$

We can assume here that $C + \varepsilon < 0$.

Let $|z - w| < \delta < r$. Then

$$u_j(w) \leq \frac{1}{\pi (r+\delta)^2} \iint_{|\zeta - w| \leq r+\delta} u_j(\zeta)\, d\lambda(\zeta).$$

Here, $\{\zeta : |\zeta - w| \leq r + \delta\} \supseteq \{\zeta : |\zeta - z| \leq r\}$. So

$$u_j(w) \leq \frac{1}{\pi (r+\delta)^2} \underbrace{\iint_{|\zeta - z| \leq r} u_j(\zeta)\, d\lambda(\zeta)}_{\leq \pi r^2 (C + \varepsilon/2)} \leq \left(\frac{r}{r+\delta} \right)^2 (C + \varepsilon/2)$$

for $j \geq j_z$. Try to take $\delta = \mu r$ for $0 < \mu < 1$. The right hand side becomes

$$\frac{1}{(1+\mu)^2} (C + \varepsilon/2),$$

and we can take μ so this is just $C + \varepsilon$. So we can take

$$\mu = \underbrace{\left(\frac{C + \varepsilon/2}{C + \varepsilon} \right)^{1/2}}_{>1} - 1.$$

We can cover K by finitely many neighborhoods of the form $\{|z - w| < \delta\}$ for $z \in K$. $\square$

Next time, we will prove the following lemma on our road to Hartogs' theorem.

Lemma 1.1. *Let $\Omega \subseteq \mathbb{C}^n$ be open, and let u be separately holomorphic in Ω. If u is locally bounded in Ω, then $u \in C(\Omega)$ (so $u \in \operatorname{Hol}(\Omega)$).*